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On Long-Period Oscillations in Coupled Systems of Nonlinear Oscillators and Stationary Systems

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Abstract

We study a system obtained by coupling a nonlinear oscillator and a stationary system. We analyze the dynamics in a singular limit and show that asymptotically orbitally stable periodic solutions can exist. We also show how a slow oscillation can arise in the coupled system.

KEYWORDS: coupled systems, relaxation oscillation, singular perturbation.

1. Introduction

We suppose that $x_1 \in \mathbf{R}^n$ denote the concentrations of substances in some unit, and that its evolution is governed by the following n-dimensional system when the unit is isolated.

$$\frac{dx}{dt} = f(x) \tag{1}$$

When the substances in the unit permeate into the medium, the time evolution is governed by the following system

$$\begin{aligned} \frac{dx_1}{dt} &= f(x_1) + \delta P(x_0 - x_1), \\ \frac{dx_0}{dt} &= \epsilon \delta P(x_1 - x_0). \end{aligned}$$

Here P is an $n \times n$ constant matrix of permeability coefficients. The parameter δ measures the strength of coupling between the unit and the medium, and $\epsilon = V_1/V_0$. Here, V_1 is the volume of the unit and V_0 is the volume of the medium. $x_1 \in \mathbf{R}^n$ represents the concentrations of the substances in the medium [3], [4]. In this case we consider the case in which (1) is a particular planar system of the form

$$\begin{aligned} \kappa \frac{dv}{dt} &= g(v,w), \\ \frac{dw}{dt} &= h(v,w), \end{aligned} \tag{2}$$

where κ is a positive parameter. We also consider the case in which only the second component communicates with the medium, and study the following system in which the coupling between the variables is generalized.

$$\begin{aligned} \kappa \frac{dv}{dt} &= g(v,w), \\ \frac{dw}{dt} &= h(v,w) + \delta p(w,x), \\ \frac{dx}{dt} &= -\sigma p(w,x). \end{aligned} \tag{3}$$

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Here we assume that δ and σ are positive parameters. It is a coupled system of (2) and the system

$$\begin{aligned} \frac{du}{dt} &= 0, \\ \frac{dx}{dt} &= 0, \end{aligned}$$

whose dynamics is stationary.

We make some assumptions under which (2) is oscillatory. Then we analyze (3) in the singular limit $\kappa \longrightarrow 0$, and show that it has an asymptotically orbitally stable periodic solution for all sufficiently large δ. We also analyze the dependence of the periods of the periodic solutions on the parameters.

2. Reduction in the Singular Limit

In this section we make some assumptions, and derive some system which govern the dynamics of (3) in the singular limit $\kappa \longrightarrow 0$. We suppose that the v-nullcline, $g(v,w) = 0$, is given by the smooth function $w = \alpha(v)$. We also suppose that there are v_ℓ and v_r such that

$$\alpha'(v) \begin{cases} < 0, & v < v_\ell, \\ > 0, & v_\ell < v < v_r, \\ < 0, & v > v_r. \end{cases}$$

We set $w_\ell = \alpha(v_\ell)$ and $w_r = \alpha(v_r)$. Then $w = \alpha(v)$ can have the inverse functions $v = \zeta_\ell(w)$ defined for $w \geq w_\ell$ and $v = \zeta_r(w)$ defined for $w \leq w_r$. We assume that $g_w(v, \alpha(v)) < 0$ for all $v \in \mathbf{R}$. We also assume that $g_{vv}(v_\ell, w_\ell) \neq 0$ and that $g_{vv}(v_r, w_r) \neq 0$. We also suppose that the w-nullcline, $h(v,w)$, is given by the smooth function $w = \beta(v)$. We assume that $h_v(v, \beta(v)) > 0$ for all $v \in \mathbf{R}$. We also assume that $h_w(v, \beta(v)) < 0$ for all $v \in \mathbf{R}$.

Then $\beta'(v) > 0$ for all $v \in \mathbf{R}$, and there is at least one point at which the v-nullcline and w-nullcline intersect. We assume that the nullclines intersect exactly at one point whose v-coordinate lies between v_ℓ and v_r. The following system is derived from the cubic BVP equations introduced in [1].

$$\begin{aligned} \kappa \frac{dv}{dt} &= v - \frac{1}{3c^2}(v-d)^3 - w + a, \\ \frac{dw}{dt} &= v - bw. \end{aligned} \tag{4}$$

Here,

$$\begin{aligned} g(v,w) &= v - \frac{1}{3c^2}(v-d)^3 - w + a, \\ h(v,w) &= v - bw. \end{aligned} \tag{5}$$

It is an example of a system that satisfies all the assumptions provided appropriate values are set for the parameters a, b, c, and d. Figure 1 shows the nullclines of (4) with

$$a = 2, \quad b = 0.4, \quad c = 0.5, \quad d = 1.5. \tag{6}$$

We denote by Γ_ℓ and Γ_r the segments of the v-nullcline defined by $\Gamma_\ell = \{(\zeta_\ell(w), w) \,|\, w_\ell \leq w \leq w_r\}$ and $\Gamma_r = \{(\zeta_r(w), w) \,|\, w_\ell \leq w \leq w_r\}$. We also denote by Q_1 and Q_2 the ends of Γ_ℓ given by $Q_1 = (\zeta_\ell(w_r), w_r)$ and $Q_2 = (v_\ell, w_\ell)$, and denote by Q_3 and Q_4 the ends of Γ_r given by $Q_3 = (\zeta_r(w_\ell), w_\ell)$ and $Q_4 = (v_r, w_r)$. Under these assumptions imposed on the functions $g(v,w)$ and $h(v,w)$, (2) has an asymptotically orbitally stable periodic solution that oscillate between a neighborhood of Γ_ℓ and a neighborhood of Γ_r when $\kappa > 0$ is sufficiently small [2]. Its period is close to the period T of the periodic solution in the singular limit $\kappa \longrightarrow 0$.

$$T = \int_{w_r}^{w_\ell} \frac{dw}{h(\zeta_\ell(w), w)} + \int_{w_\ell}^{w_r} \frac{dw}{h(\zeta_r(w), w)}.$$

In fact, any solution eventually oscillates between a neighborhood of Γ_ℓ and a neighborhood of Γ_r. A solution, which starts at an initial point in a neighborhood of Γ_ℓ, varies slowly staying close to Γ_ℓ, and

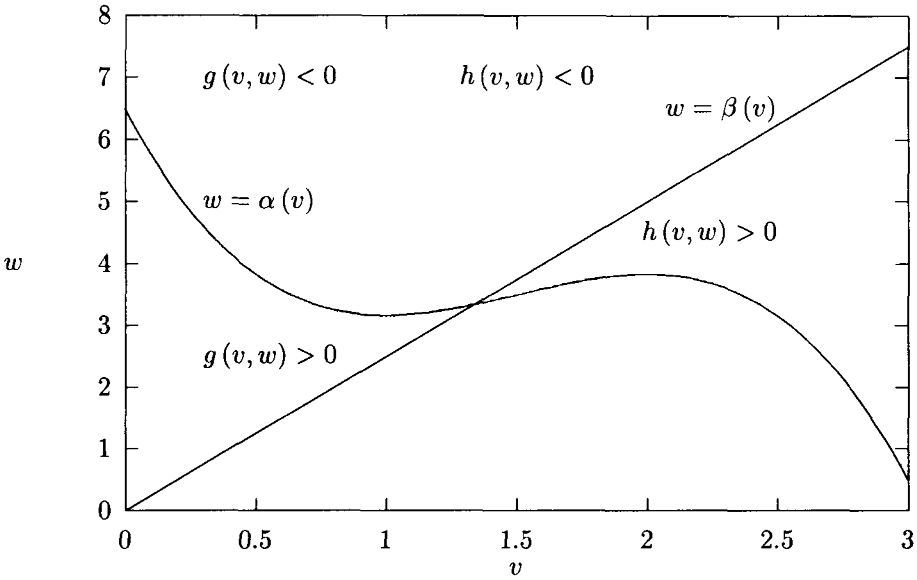

Figure 1: The nullclines of (4). The nullclines $w = \alpha(v)$ and $w = \beta(v)$ of (4), in which (5) and (6) are substituted, are shown.

travels toward Q_2. When it becomes sufficiently close to Q_2, it starts varying fast, and it travels almost horizontally toward Q_3. Then it starts varying slowly again, and travels toward Q_4 staying close to Γ_r. When it becomes sufficiently close to Q_4, the solution starts varying fast, and then travels almost horizontally toward Q_1. Then it returns in a neighborhood of Γ_ℓ. Such behavior of the solutions is called relaxation oscillation.

We assume that the function $p(w,x)$ is a smooth function. We also assume that $p_w(w,x) < 0$ for all $(w,x) \in \mathbf{R}^2$, and that $p_x(w,x) > 0$ for all $(w,x) \in \mathbf{R}^2$. We suppose that the x-nullcline is given by a smooth function $x = \eta(w)$. Then It follows that $\eta'(w) > 0$ for all $w \in \mathbf{R}$ and that

$$p(w,x)\begin{cases} <0, & x<\eta(w), \\ >0, & x>\eta(w). \end{cases}$$

In the singular limit $\kappa \longrightarrow 0$, solutions of (3) are restricted to the regions Ω_ℓ and Ω_r defined by

$$\Omega_\ell = \{(v,w,x)\,|v=\zeta_\ell(w), w \geq w_\ell\}, \quad \Omega_r = \{(v,w,x)\,|v=\zeta_r(w), w \leq w_r\}.$$

In Ω_ℓ, the dynamics is governed by the following system.

$$\begin{aligned} \frac{dw}{dt} &= h(\zeta_\ell(w), w) + \delta p(w,x), \\ \frac{dx}{dt} &= -\sigma p(w,x). \end{aligned} \tag{7}$$

This system is valid in the region Λ_ℓ defined by $\Lambda_\ell = \{(w,x)\,|\ w \geq w_\ell\}$.

We suppose that w-nullcline is given by a smooth function $x = \xi_\ell(w)$. Then $\xi_\ell(w) > \eta(w)$ and $\xi_\ell'(w) > 0$ for all $w \in \mathbf{R}$ Moreover,

$$h(\zeta_\ell(w), w) + \delta p(w,x)\begin{cases} <0, & x<\xi_\ell(w), \\ >0, & x>\xi_\ell(w). \end{cases}$$

(5) and (6) are substituted in (2), and

$$p(w,x) = x - w, \tag{8}$$

and

$$\sigma = \epsilon\delta, \quad \delta = 1, \quad \epsilon = 10 \tag{9}$$

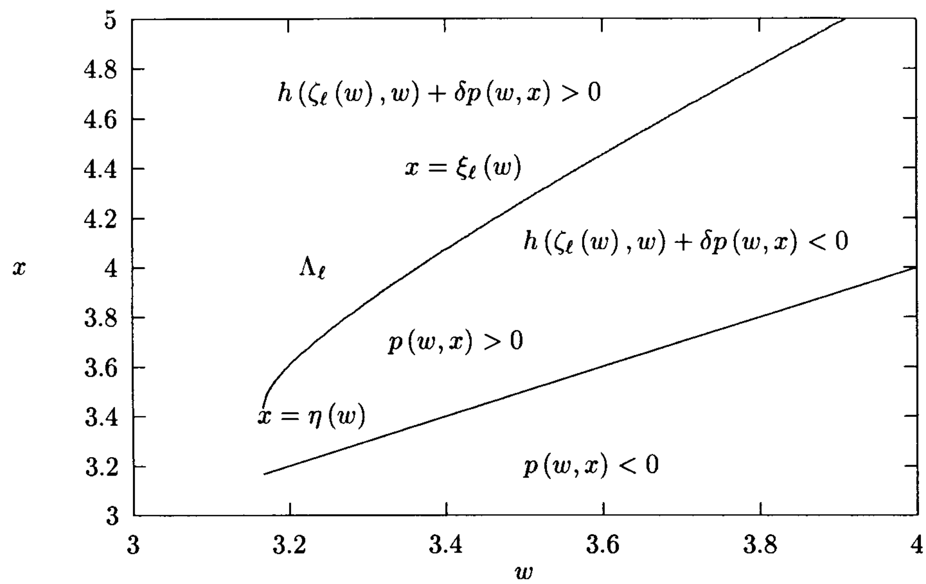

Figure 2: The nullclines of the equations on Ω_ℓ.

are substituted in (3). Then the nullclines of (7) are numerically generated and they are shown in Figure 2. In Ω_r, the dynamics is governed by the following system.

$$\begin{aligned} \frac{dw}{dt} &= h(\zeta_r(w), w) + \delta p(w, x), \\ \frac{dx}{dt} &= -\sigma p(w, x). \end{aligned} \tag{10}$$

This system is valid in the region $(w, x) \in \Lambda_r$ defined by $\Lambda_r = \{(w, x) \mid w \le w_r\}$. We also suppose that the w-nullcline is given by a smooth function $x = \xi_r(w)$. Then $\xi_r(w) < \eta(w)$ and $\xi_r'(w) > 0$ for all $w \in \mathbf{R}$, and

$$h(\zeta_r(w), w) + \delta p(w, x) \begin{cases} < 0, & x < \xi_r(w), \\ > 0, & x > \xi_r(w). \end{cases}$$

(5) and (6) are substituted in (2), and (8) and (9) are substituted in (3). Then the nullclines of (10) are numerically generated and they are shown in Figure 3.

In the singular limit $\kappa \longrightarrow 0$, solutions of (3) can be constructed from solutions of (7) and solutions of (10). We denote by $\phi(t, P_0) = (v(t), w(t), x(t))$ the solution of (3) with the initial value $\phi(0, P_0) = P_0 = (v_0, w_0, x_0)$. We also denote by $\phi_\ell(t, Q_0)$ the solution of (7) with the initial value $\phi_\ell(t, Q_0) = Q_0 = (w_0, x_0)$, and by $\phi_r(t, Q_0)$ the solution of (10) with the initial value $\phi_r(0, Q_0) = Q_0$. Suppose that $v_0 = \zeta_\ell(w_0)$. Then $P_0 \in \Omega_\ell$, and $v(t) = \zeta_\ell(w(t))$ and $(w(t), x(t)) = \phi_\ell(t, Q_0)$ as long as $\phi_\ell(t, Q_0) \in \Lambda_\ell$. However, $\phi_\ell(t, Q_0)$ eventually reaches the boundary of Λ_ℓ, the line $w = w_\ell$. Suppose that $\phi_\ell(t, Q_0)$ reaches the line $w = w_\ell$ at $t = t_1$. Let $Q_1 = (w_\ell, x_1) = \phi_\ell(t_1, Q_0)$. Then $\phi(t, P_0)$ jumps to the point $P_1 = (\zeta_r(w_\ell), w_\ell, x_1)$ at $t = t_1$, and for $t \ge t_1$, $v(t) = \zeta_r(w(t))$ $(w(t), x(t)) = \phi_r(t - t_1, P_1)$ as long as $\phi_r(t - t_1, P_1) \in \Lambda_r$. Here, we define a function Φ_ℓ on Λ_ℓ by

$$\Phi_\ell(Q_0) = x_1.$$

On the other hand, If $v_0 = \zeta_r(w_0)$, then $P_0 \in \Omega_r$. In this case, $v(t) = \zeta_r(w(t))$ and $(w(t), x(t)) = \phi_r(t, Q_0)$ as long as $\phi_r(t, Q_0) \in \Lambda_r$. $\phi_r(t, Q_0)$ eventually reaches the boundary of Λ_r, the line $w = w_r$. Suppose that $\phi_r(t, Q_0)$ reaches line $w = w_r$ at $t = t_1$. let $Q_1 = (w_r, x_1) = \phi_r(t_1, Q_0)$. Then $\phi(t, P_0)$ jumps to the point $P_1 = (\zeta_\ell(w_r), w_r, x_1)$ at $t = t_1$, and for $t \ge t_1$, $v(t) = \zeta_\ell(w(t))$ and $(w(t), x(t)) = \phi_\ell(t - t_1, Q_1)$. We define a function Φ_r on Λ_r by

$$\Phi_r(Q_0) = x_1.$$

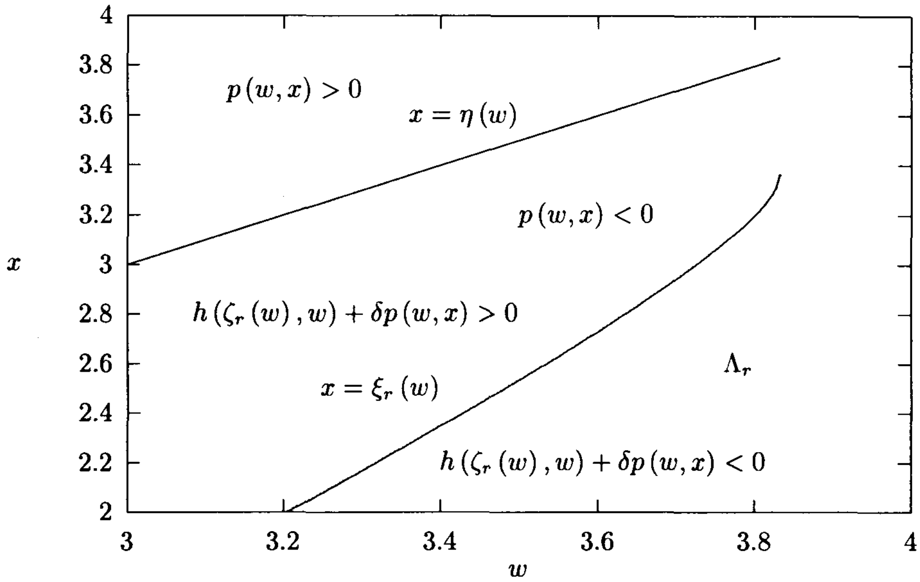

Figure 3: The nullclines of the equations in Ω_r.

3. Periodic Solutions in the Singular Limit

In this section, we define a tow-parameter family of maps whose fixed points give rise to periodic solutions of (3). We show that a map has a unique fixed point for all sufficiently large δ, and analyze the stability of the periodic solutions.

we define a two-parameter family of maps Φ by

$$\Phi(x_0) = \Phi_r(w_\ell, \Phi_\ell(w_r, x_0)).$$

Suppose that x_0 is a fixed point of Φ, and let $P_0 = (\zeta_\ell(w_r), w_r, x_0)$. Then it is easily seen that $\phi(t, P_0)$ is a periodic solution of (3). We show that ϕ has a unique fixed point for all sufficiently large δ. It is easily seen that the region bounded by the nullclines $x = \xi_\ell(w)$ and $x = \eta(w)$ is an invariant region of (7). Moreover Φ_ℓ maps the half line $\{(w_r, x) \,|x \geq a\}$ into the interval $[\eta(w_\ell), \xi_\ell(w_\ell)]$ when $a \geq \eta(w_\ell)$. Similarly, the region bounded by the nullclines $x = \xi_r(w)$ and $x = \eta(w)$ is an invariant region of (10), and Φ_r maps the half line $\{(w_\ell, x) \,|x \leq b\}$ into the interval $[\xi_r(w_r), \eta(w_r)]$ when $b \leq \eta(w_r)$. We note that $\xi_\ell(w) \longrightarrow \eta(w)$ as $\delta \longrightarrow \infty$, and that $\xi_r(w) \longrightarrow \eta(w)$ as $\delta \longrightarrow \infty$. Then there is a δ^* such that $\xi_r(w_r) > \eta(w_\ell)$, $\xi_\ell(w_\ell) < \eta(w_r)$ Then for $\delta > \delta^*$, Φ maps the interval $[\eta(w_r), \xi_r(w_r)]$ into itself. Since $\Phi(w)$ is a continuous function, it has a fixed point. Next, we analyze the stability of periodic solutions corresponding to fixed points of Φ. Let $x = \psi_\ell(w, w_0, x_0)$ be the solution of

$$\frac{dx}{dw} = X_\ell(w,x) \equiv -\frac{\sigma p(w,x)}{h(\zeta_\ell(w), w) + \delta p(w,x)}$$

with the initial value $\psi_\ell(w_0, w_0, x_0) = x_0$, and let $x = \psi_r(w, w_0, x_0)$ be the solution of

$$\frac{dx}{dw} = X_r(w,x) \equiv -\frac{\sigma p(w,x)}{h(\zeta_r(w), w) + \delta p(w,x)}$$

with the initial value $\psi_r(w_0, w_0, x_0) = x_0$. Then if $x_0 \in [\eta(w_r), \xi_r(w_r)]$, $\Phi(x_0) = \psi_r(w_r, w_\ell, \psi_\ell(w_\ell, w_r, x_0))$, and

$$\Phi'(x_0) = \frac{\partial \psi_r}{\partial x_0}(w_r, w_\ell, \psi_\ell(w_\ell, w_r, x_0)) \frac{\partial \psi_\ell}{\partial x_0}(w_\ell, w_r, x_0).$$

Since

$$\frac{\partial X_\ell}{\partial x} = -\frac{\sigma p_x(w,x)\, h(\zeta_\ell(w), w)}{[h(\zeta_\ell(w), w) + \delta p(w,x)]^2} > 0,$$

$$0 < \frac{\partial \psi_\ell}{\partial x_0}(w_\ell, w_r, x_0) < 1.$$

Similarly,

$$\frac{\partial X_r}{\partial x} = -\frac{\sigma p_x(w,x)\, h(\zeta_r(w), w)}{[h(\zeta_r(w), w) + \delta p(w,x)]^2} > 0,$$

and this implies

$$0 < \frac{\partial \psi_r}{\partial x_0}(w_r, w_\ell, x_0) < 1.$$

It follows that $0 < \Phi'(x_0) < 1$. We summarize theses results in the following proposition.

Proposition 1 *There is a $\delta^* > 0$ such that the two-parameter family of the maps Φ has a unique fixed point $x_0 = x_0(\delta, \sigma)$ for every fixed pair (δ, σ) provided $\delta > \delta^*$. Moreover, $0 < \Phi'(x_0(\delta, \sigma)) < 1$.*

Proposition 1 guarantees the existence of a two-parameter family of periodic solutions. Moreover, it shows that the the periodic solution corresponding to the fixed point of Φ is asymptotically orbitally stable for every fixed pair (δ, σ) with $\delta > \delta^*$. In that case, A result in [2] indicates that (3) has at least one periodic solution near the periodic solution in the singular limit provided $\kappa > 0$ is sufficiently small.

Now suppose that the parameter σ is given by $\sigma = \epsilon\delta$. Define $y = \epsilon w + x$, $z = w - x$. Then (7) becomes

$$\begin{aligned} \frac{dy}{dt} &= \epsilon h\left(\zeta_\ell\left(\frac{y+z}{1+\epsilon}\right), \frac{y+z}{1+\epsilon}\right), \\ \frac{dz}{dt} &= h\left(\zeta_\ell\left(\frac{y+z}{1+\epsilon}\right), \frac{y+z}{1+\epsilon}\right) + (1+\epsilon)\delta p\left(\frac{y+z}{1+\epsilon}, \frac{y-\epsilon z}{1+\epsilon}\right). \end{aligned} \tag{11}$$

In particular, when $p(w,x) = x - w$, one expects that $z \longrightarrow 0$ as $\delta \longrightarrow \infty$. Then, we obtain the following equation from (11).

$$\frac{dy}{dt} = \epsilon h\left(\zeta_\ell\left(\frac{y}{1+\epsilon}\right), \frac{y}{1+\epsilon}\right).$$

Similarly, one obtains the following equation from (10).

$$\frac{dy}{dt} = \epsilon h\left(\zeta_\ell\left(\frac{y}{1+\epsilon}\right), \frac{y}{1+\epsilon}\right).$$

It follows that the period of the periodic solution is close to

$$T_\epsilon = \int_{(1+\epsilon)w_r}^{(1+\epsilon)w_\ell} \frac{dy}{\epsilon h\left(\zeta_\ell\left(\frac{y}{1+\epsilon}\right), \frac{y}{1+\epsilon}\right)} + \int_{(1+\epsilon)w_\ell}^{(1+\epsilon)w_r} \frac{dy}{\epsilon h\left(\zeta_r\left(\frac{y}{1+\epsilon}\right), \frac{y}{1+\epsilon}\right)} = \left(1 + \epsilon^{-1}\right) T.$$

Therefore the increment of the period by coupling is proportional to the relative capacity of the medium ϵ^{-1}, and a long-period oscillation can occur.

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